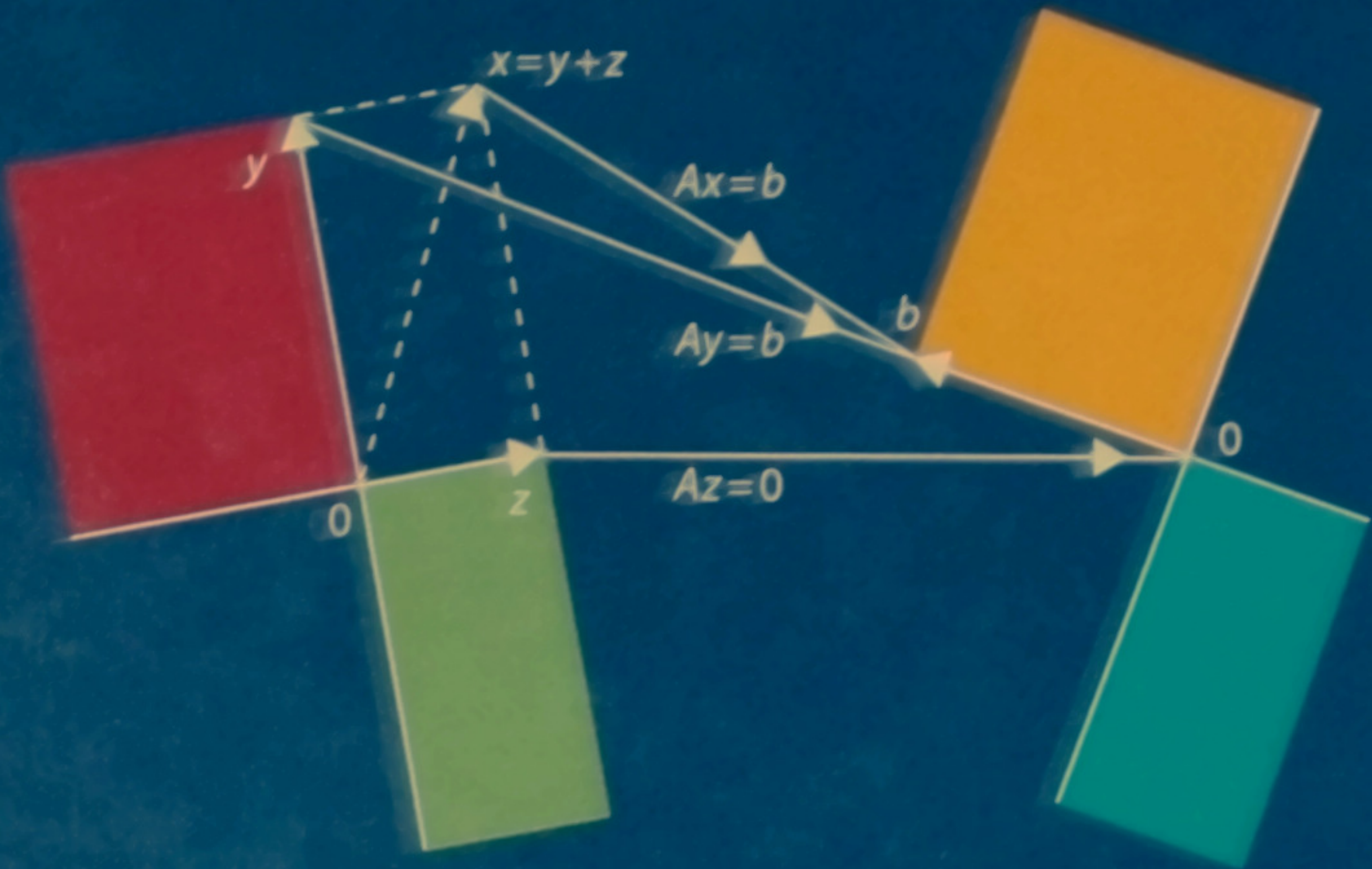
Introduction to
LINEAR ALGEBRA

FOURTH EDITION



GILBERT STRANG

INTRODUCTION
TO
LINEAR ALGEBRA

Fourth Edition

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The website for this book is **math.mit.edu/linearalgebra**.
A Solutions Manual is available to instructors by email from the publisher.

Course material including syllabus and Teaching Codes and exams and also videotaped lectures are available on the teaching website: **web.mit.edu/18.06**

Linear Algebra is included in MIT's OpenCourseWare site **ocw.mit.edu**.
This provides video lectures of the full linear algebra course 18.06.
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The front cover captures a central idea of linear algebra.
$A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the (orange) column space of A.
One particular solution $\boldsymbol{y}$ is in the (red) row space: $A\boldsymbol{y} = \boldsymbol{b}$.
Add any vector $\boldsymbol{z}$ from the (green) nullspace of A: $A\boldsymbol{z} = \boldsymbol{0}$.
The complete solution is $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$. Then $A\boldsymbol{x} = A\boldsymbol{y} + A\boldsymbol{z} = \boldsymbol{b}$.

The cover design was the inspiration of a creative collaboration:
Lois Sellers (**birchdesignassociates.com**) and Gail Corbett.

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Preface

I will be happy with this preface if three important points come through clearly:

1. The beauty and variety of linear algebra, and its extreme usefulness
2. The goals of this book, and the new features in this Fourth Edition
3. The steady support from our linear algebra websites and the video lectures

May I begin with notes about two websites that are constantly used, and the new one.

ocw.mit.edu Messages come from thousands of students and faculty about linear algebra on this OpenCourseWare site. The 18.06 course includes video lectures of a complete semester of classes. Those lectures offer an independent review of the whole subject based on this textbook—the professor's time stays free and the student's time can be 3 a.m. (The reader doesn't have to be in a class at all.) A million viewers around the world have seen these videos (*amazing*). I hope you find them helpful.

web.mit.edu/18.06 This site has homeworks and exams (with solutions) for the current course as it is taught, and as far back as 1996. There are also review questions, Java demos, Teaching Codes, and short essays (*and the video lectures*). My goal is to make this book as useful as possible, with all the course material we can provide.

math.mit.edu/linearalgebra The newest website is devoted specifically to this Fourth Edition. It will be a permanent record of ideas and codes and good problems and solutions. Several sections of the book are directly available online, plus notes on teaching linear algebra. The content is growing quickly and contributions are welcome from everyone.

The Fourth Edition

Thousands of readers know earlier editions of *Introduction to Linear Algebra*. The new cover shows the **Four Fundamental Subspaces**—the row space and nullspace are on the left side, the column space and the nullspace of A^{T} are on the right. It is not usual to put the central ideas of the subject on display like this! You will meet those four spaces in Chapter 3, and you will understand why that picture is so central to linear algebra.

Those were named the Four Fundamental Subspaces in my first book, and they start from a matrix A. Each row of A is a vector in n-dimensional space. When the matrix has m rows, each column is a vector in m-dimensional space. The crucial operation in linear algebra is taking ***linear combinations of vectors***. (That idea starts on page 1 of the book and never stops.) *When we take all linear combinations of the column vectors, we get the column space.* If this space includes the vector b, we can solve the equation $Ax = b$.

I have to stop here or you won't read the book. May I call special attention to the new Section 1.3 in which these ideas come early—with two specific examples. You are not expected to catch every detail of vector spaces in one day! But you will see the first matrices in the book, and a picture of their column spaces, and even an *inverse matrix*. You will be learning the language of linear algebra in the best and most efficient way: by using it.

Every section of the basic course now ends with ***Challenge Problems***. They follow a large collection of review problems, which ask you to use the ideas in that section—the dimension of the column space, a basis for that space, the rank and inverse and determinant and eigenvalues of A. Many problems look for computations by hand on a small matrix, and they have been highly praised. The new Challenge Problems go a step further, and sometimes they go deeper. Let me give four examples:

Section 2.1: Which row exchanges of a Sudoku matrix produce another Sudoku matrix?

Section 2.4: From the shapes of A, B, C, is it faster to compute AB times C or A times BC?

Background: The great fact about multiplying matrices is that *AB times C gives the same answer as A times BC*. This simple statement is the reason behind the rule for matrix multiplication. If AB is square and C is a vector, it's faster to do BC first. Then multiply by A to produce ABC. The question asks about other shapes of A, B, and C.

Section 3.4: If $Ax = b$ and $Cx = b$ have the same solutions for every b, is $A = C$?

Section 4.1: What conditions on the four vectors r, n, c, ℓ allow them to be bases for the row space, the nullspace, the column space, and the left nullspace of a 2 by 2 matrix?

The Start of the Course

The equation $Ax = b$ uses the language of linear combinations right away. The vector Ax is *a combination of the columns of A*. The equation is asking for ***a combination that produces b***. The solution vector x comes at three levels and all are important:

1. ***Direct solution*** to find x by forward elimination and back substitution.

2. ***Matrix solution*** using the inverse of A: $x = A^{-1}b$ (if A has an inverse).

3. ***Vector space solution*** $x = y + z$ as shown on the cover of the book:

   Particular solution (to $Ay = b$) plus ***nullspace solution*** (to $Az = 0$)

Direct elimination is the most frequently used algorithm in scientific computing, and the idea is not hard. Simplify the matrix A so it becomes triangular—then all solutions come quickly. I don't spend forever on practicing elimination, it will get learned.

The speed of every new supercomputer is tested on $Ax = b$: it's pure linear algebra. IBM and Los Alamos announced a new world record of 10^{15} operations per second in 2008.

That *petaflop speed* was reached by solving many equations in parallel. High performance computers avoid operating on single numbers, they feed on whole submatrices.

The processors in the Roadrunner are based on the Cell Engine in PlayStation 3. What can I say, video games are now the largest market for the fastest computations.

Even a supercomputer doesn't want the inverse matrix: too slow. Inverses give the simplest formula $x = A^{-1}b$ but not the top speed. And everyone must know that determinants are even slower—there is no way a linear algebra course should begin with formulas for the determinant of an n by n matrix. Those formulas have a place, but not first place.

Structure of the Textbook

Already in this preface, you can see the style of the book and its goal. That goal is serious, to explain this beautiful and useful part of mathematics. You will see how the applications of linear algebra reinforce the key ideas. I hope every teacher will learn something new; familiar ideas can be seen in a new way. The book moves gradually and steadily from *numbers* to *vectors* to *subspaces*—each level comes naturally and everyone can get it.

Here are ten points about the organization of this book:

1. Chapter 1 starts with vectors and dot products. If the class has met them before, focus quickly on linear combinations. The new Section 1.3 provides three independent vectors whose combinations fill all of 3-dimensional space, and three dependent vectors in a plane. ***Those two examples are the beginning of linear algebra.***

2. Chapter 2 shows the row picture and the column picture of $Ax = b$. The heart of linear algebra is in that connection between the rows of A and the columns: the same numbers but very different pictures. Then begins the algebra of matrices: an elimination matrix E multiplies A to produce a zero. The goal here is to capture the whole process—start with A and end with an ***upper triangular*** U.

   Elimination is seen in the beautiful form $A = LU$. The ***lower triangular*** L holds all the forward elimination steps, and U is the matrix for back substitution.

3. Chapter 3 is linear algebra at the best level: ***subspaces***. The column space contains all linear combinations of the columns. The crucial question is: ***How many of those columns are needed?*** The answer tells us the dimension of the column space, and the key information about A. We reach the Fundamental Theorem of Linear Algebra.

4. Chapter 4 has m equations and only n unknowns. It is almost sure that $Ax = b$ has no solution. We cannot throw out equations that are close but not perfectly exact. When we solve by ***least squares***, the key will be the matrix $A^{\mathrm{T}}A$. This wonderful matrix $A^{\mathrm{T}}A$ appears everywhere in applied mathematics, when A is rectangular.

5. ***Determinants*** in Chapter 5 give formulas for all that has come before—inverses, pivots, volumes in n-dimensional space, and more. We don't need those formulas to compute! They slow us down. But $\det A = 0$ tells when a matrix is singular, and that test is the key to eigenvalues.

6. ***Section 6.1 introduces eigenvalues for 2 by 2 matrices.*** Many courses want to see eigenvalues early. It is completely reasonable to come here directly from Chapter 3, because the determinant is easy for a 2 by 2 matrix. *The key equation is* $Ax = \lambda x$.

   Eigenvalues and eigenvectors are an astonishing way to understand a square matrix. They are not for $Ax = b$, they are for dynamic equations like $du/dt = Au$. The idea is always the same: *follow the eigenvectors.* In those special directions, A acts like a single number (the eigenvalue λ) and the problem is one-dimensional.

   Chapter 6 is full of applications. One highlight is ***diagonalizing a symmetric matrix.*** Another highlight—not so well known but more important every day—is the diagonalization of *any matrix.* This needs two sets of eigenvectors, not one, and they come (of course!) from $A^{\mathrm{T}}A$ and AA^{T}. This Singular Value Decomposition often marks the end of the basic course and the start of a second course.

7. Chapter 7 explains the ***linear transformation*** approach—it is linear algebra without coordinates, the ideas without computations. Chapter 9 is the opposite—all about how $Ax = b$ and $Ax = \lambda x$ are really solved. Then Chapter 10 moves from real numbers and vectors to complex vectors and matrices. The Fourier matrix F is the most important complex matrix we will ever see. And the ***Fast Fourier Transform*** (multiplying quickly by F and F^{-1}) is a revolutionary algorithm.

8. Chapter 8 is full of applications, more than any single course could need:

   8.1 *Matrices in Engineering*—differential equations replaced by matrix equations

   8.2 *Graphs and Networks*—leading to the edge-node matrix for Kirchhoff's Laws

   8.3 *Markov Matrices*—as in Google's *PageRank* algorithm

   8.4 *Linear Programming*—a new requirement $x \geq 0$ and minimization of the cost

   8.5 *Fourier Series*—linear algebra for functions and digital signal processing

   8.6 *Matrices in Statistics and Probability*—$Ax = b$ is weighted by average errors

   8.7 *Computer Graphics*—matrices move and rotate and compress images.

9. Every section in the basic course ends with a ***Review of the Key Ideas.***

10. How should computing be included in a linear algebra course? It can open a new understanding of matrices—every class will find a balance. I chose the language of MATLAB as a direct way to describe linear algebra: eig(ones(4)) will produce the eigenvalues 4, 0, 0, 0 of the 4 by 4 all-ones matrix. *Go to **netlib.org** for codes.*

    You can freely choose a different system. More and more software is open source.

The new website ***math.mit.edu/linearalgebra*** provides further ideas about teaching and learning. Please contribute! Good problems are welcome by email: ***gs@math.mit.edu.*** Send new applications too, linear algebra is an incredibly useful subject.

The Variety of Linear Algebra

Calculus is mostly about one special operation (the derivative) and its inverse (the integral). Of course I admit that calculus could be important .... But so many applications of mathematics are discrete rather than continuous, digital rather than analog. The century of data has begun! You will find a light-hearted essay called "Too Much Calculus" on my website. ***The truth is that vectors and matrices have become the language to know.***

Part of that language is the wonderful variety of matrices. Let me give three examples:

Symmetric matrix	***Orthogonal matrix***	***Triangular matrix***
$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A key goal is learning to "read" a matrix. You need to see the meaning in the numbers. This is really the essence of mathematics—patterns and their meaning.

May I end with this thought for professors. You might feel that the direction is right, and wonder if your students are ready. ***Just give them a chance***! Literally thousands of students have written to me, frequently with suggestions and surprisingly often with thanks. They know this course has a purpose, because the professor and the book are on their side. Linear algebra is a fantastic subject, enjoy it.

Help With This Book

I can't even name all the friends who helped me, beyond thanking Brett Coonley at MIT and Valutone in Mumbai and SIAM in Philadelphia for years of constant and dedicated support. The greatest encouragement of all is the feeling that you are doing something worthwhile with your life. Hundreds of generous readers have sent ideas and examples and corrections (and favorite matrices!) that appear in this book. *Thank you all.*

Background of the Author

This is my eighth textbook on linear algebra, and I have not written about myself before. I hesitate to do it now. It is the mathematics that is important, and the reader. The next paragraphs add something personal as a way to say that textbooks are written by people.

I was born in Chicago and went to school in Washington and Cincinnati and St. Louis. My college was MIT (and my linear algebra course was *extremely abstract*). After that came Oxford and UCLA, then back to MIT for a very long time. I don't know how many thousands of students have taken 18.06 (more than a million when you include the videos on ***ocw.mit.edu***). The time for a fresh approach was right, because this fantastic subject was only revealed to math majors—**we needed to open linear algebra to the world.**

Those years of teaching led to the Haimo Prize from the Mathematical Association of America. For encouraging education worldwide, the International Congress of Industrial and Applied Mathematics awarded me the first Su Buchin Prize. I am extremely grateful, more than I could possibly say. What I hope most is that you will like linear algebra.

Chapter 1

Introduction to Vectors

The heart of linear algebra is in two operations—both with vectors. We add vectors to get $\boldsymbol{v}+\boldsymbol{w}$. We multiply them by numbers c and d to get $c\boldsymbol{v}$ and $d\boldsymbol{w}$. Combining those two operations (adding $c\boldsymbol{v}$ to $d\boldsymbol{w}$) gives the ***linear combination*** $c\boldsymbol{v}+d\boldsymbol{w}$.

Linear combination $$c\boldsymbol{v}+d\boldsymbol{w}=c\begin{bmatrix}1\\1\end{bmatrix}+d\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}c+2d\\c+3d\end{bmatrix}$$

Example $$\boldsymbol{v}+\boldsymbol{w}=\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}\text{ is the combination with } c=d=1$$

Linear combinations are all-important in this subject! Sometimes we want one particular combination, the specific choice $c=2$ and $d=1$ that produces $c\boldsymbol{v}+d\boldsymbol{w}=(4,5)$. Other times we want *all the combinations* of $\boldsymbol{v}$ and $\boldsymbol{w}$ (coming from all c and d).

The vectors $c\boldsymbol{v}$ lie along a line. When $\boldsymbol{w}$ is not on that line, **the combinations** $c\boldsymbol{v}+d\boldsymbol{w}$ **fill the whole two-dimensional plane**. (I have to say "two-dimensional" because linear algebra allows higher-dimensional planes.) Starting from four vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{z}$ in four-dimensional space, their combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}+f\boldsymbol{z}$ are likely to fill the space—but not always. The vectors and their combinations could even lie on one line.

Chapter 1 explains these central ideas, on which everything builds. We start with two-dimensional vectors and three-dimensional vectors, which are reasonable to draw. Then we move into higher dimensions. The really impressive feature of linear algebra is how smoothly it takes that step into n-dimensional space. Your mental picture stays completely correct, even if drawing a ten-dimensional vector is impossible.

This is where the book is going (into n-dimensional space). The first steps are the operations in Sections 1.1 and 1.2. Then Section 1.3 outlines three fundamental ideas.

1.1 ***Vector addition*** $\boldsymbol{v}+\boldsymbol{w}$ ***and linear combinations*** $c\boldsymbol{v}+d\boldsymbol{w}$.

1.2 ***The dot product*** $\boldsymbol{v}\cdot\boldsymbol{w}$ ***of two vectors and the length*** $\|\boldsymbol{v}\|=\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$.

1.3 ***Matrices*** A, ***linear equations*** $A\boldsymbol{x}=\boldsymbol{b}$, ***solutions*** $\boldsymbol{x}=A^{-1}\boldsymbol{b}$.

1.1 Vectors and Linear Combinations

"You can't add apples and oranges." In a strange way, this is the reason for vectors. We have two separate numbers v_1 and v_2. That pair produces a ***two-dimensional vector v***:

$$\textbf{Column vector} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{matrix} v_1 = \text{first component} \\ v_2 = \text{second component} \end{matrix}$$

We write $\boldsymbol{v}$ as a ***column***, not as a row. The main point so far is to have a single letter $\boldsymbol{v}$ (in ***boldface italic***) for this pair of numbers v_1 and v_2 (in *lightface italic*).

Even if we don't add v_1 to v_2, we do *add vectors*. The first components of $\boldsymbol{v}$ and $\boldsymbol{w}$ stay separate from the second components:

$$\textbf{VECTOR ADDITION} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \text{ add to } \boldsymbol{v}+\boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

You see the reason. We want to add apples to apples. Subtraction of vectors follows the same idea: *The components of* $\boldsymbol{v} - \boldsymbol{w}$ *are* $v_1 - w_1$ *and* $v_2 - w_2$.

The other basic operation is *scalar multiplication*. Vectors can be multiplied by 2 or by -1 or by any number c. There are two ways to double a vector. One way is to add $\boldsymbol{v} + \boldsymbol{v}$. The other way (the usual way) is to multiply each component by 2:

$$\textbf{SCALAR MULTIPLICATION} \qquad 2\boldsymbol{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} \text{ and } -\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

The components of $c\boldsymbol{v}$ are cv_1 and cv_2. The number c is called a "scalar".

Notice that the sum of $-\boldsymbol{v}$ and $\boldsymbol{v}$ is the zero vector. This is $\mathbf{0}$, which is not the same as the number zero! The vector $\mathbf{0}$ has components 0 and 0. Forgive me for hammering away at the difference between a vector and its components. Linear algebra is built on these operations $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$—***adding vectors and multiplying by scalars***.

The order of addition makes no difference: $\boldsymbol{v} + \boldsymbol{w}$ equals $\boldsymbol{w} + \boldsymbol{v}$. Check that by algebra: The first component is $v_1 + w_1$ which equals $w_1 + v_1$. Check also by an example:

$$\boldsymbol{v}+\boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{8} \end{bmatrix} \qquad \boldsymbol{w}+\boldsymbol{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{8} \end{bmatrix}.$$

Linear Combinations

Combining addition with scalar multiplication, we now form "*linear combinations*" of $\boldsymbol{v}$ and $\boldsymbol{w}$. Multiply $\boldsymbol{v}$ by c and multiply $\boldsymbol{w}$ by d; then add $c\boldsymbol{v} + d\boldsymbol{w}$.

DEFINITION ***The sum of $c\boldsymbol{v}$ and $d\boldsymbol{w}$ is a*** *linear combination* ***of $\boldsymbol{v}$ and $\boldsymbol{w}$.***

Four special linear combinations are: sum, difference, zero, and a scalar multiple $c\boldsymbol{v}$:

$$\begin{aligned} 1\boldsymbol{v} + 1\boldsymbol{w} &= \text{sum of vectors in Figure 1.1a} \\ 1\boldsymbol{v} - 1\boldsymbol{w} &= \text{difference of vectors in Figure 1.1b} \\ 0\boldsymbol{v} + 0\boldsymbol{w} &= \textit{zero vector} \\ c\boldsymbol{v} + 0\boldsymbol{w} &= \text{vector } c\boldsymbol{v} \text{ in the direction of } \boldsymbol{v} \end{aligned}$$

The zero vector is always a possible combination (its coefficients are zero). Every time we see a "space" of vectors, that zero vector will be included. This big view, taking *all* the combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$, is linear algebra at work.

The figures show how you can visualize vectors. For algebra, we just need the components (like 4 and 2). That vector $\boldsymbol{v}$ is represented by an arrow. The arrow goes $v_1 = 4$ units to the right and $v_2 = 2$ units up. It ends at the point whose x, y coordinates are $4, 2$. This point is another representation of the vector—so we have three ways to describe $\boldsymbol{v}$:

Represent vector $\boldsymbol{v}$ Two numbers Arrow from $(0, 0)$ Point in the plane

We add using the numbers. We visualize $\boldsymbol{v} + \boldsymbol{w}$ using arrows:

Vector addition (head to tail) ***At the end of $\boldsymbol{v}$, place the start of $\boldsymbol{w}$.***

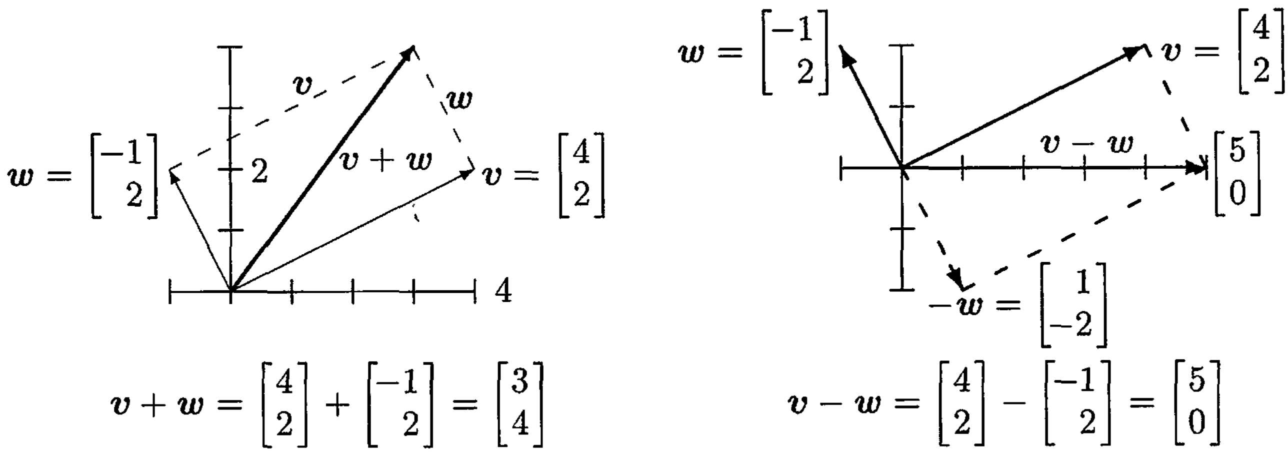


Figure 1.1: Vector addition $\boldsymbol{v} + \boldsymbol{w} = (3,\ 4)$ produces the diagonal of a parallelogram. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w} = (5,\ 0)$.

We travel along $\boldsymbol{v}$ and then along $\boldsymbol{w}$. Or we take the diagonal shortcut along $\boldsymbol{v} + \boldsymbol{w}$. We could also go along $\boldsymbol{w}$ and then $\boldsymbol{v}$. In other words, $\boldsymbol{w} + \boldsymbol{v}$ gives the same answer as $\boldsymbol{v} + \boldsymbol{w}$.

These are different ways along the parallelogram (in this example it is a rectangle). The sum is the diagonal vector $\boldsymbol{v} + \boldsymbol{w}$.

The zero vector $\mathbf{0} = (0, 0)$ is too short to draw a decent arrow, but you know that $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$. For $2\boldsymbol{v}$ we double the length of the arrow. We reverse $\boldsymbol{w}$ to get $-\boldsymbol{w}$. This reversing gives the subtraction on the right side of Figure 1.1.

Vectors in Three Dimensions

A vector with two components corresponds to a point in the xy plane. The components of $\boldsymbol{v}$ are the coordinates of the point: $x = v_1$ and $y = v_2$. The arrow ends at this point (v_1, v_2), when it starts from $(0, 0)$. Now we allow vectors to have three components (v_1, v_2, v_3).

The xy plane is replaced by three-dimensional space. Here are typical vectors (still column vectors but with three components):

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

The vector $\boldsymbol{v}$ corresponds to an arrow in 3-space. Usually the arrow starts at the "origin", where the xyz axes meet and the coordinates are $(0, 0, 0)$. The arrow ends at the point with coordinates v_1, v_2, v_3. There is a perfect match between the ***column vector*** and the ***arrow from the origin*** and the ***point where the arrow ends***.

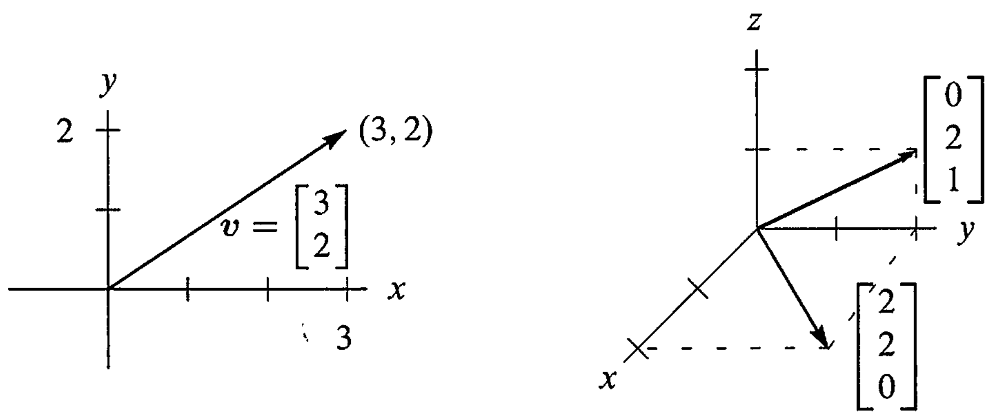


Figure 1.2: Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

From now on $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ***is also written as*** $\boldsymbol{v} = (1, 1, -1)$.

The reason for the row form (in parentheses) is to save space. But $\boldsymbol{v} = (1, 1, -1)$ is not a row vector! It is in actuality a column vector, just temporarily lying down. The row vector $[\,1 \quad 1 \quad -1\,]$ is absolutely different, even though it has the same three components. That row vector is the "transpose" of the column $\boldsymbol{v}$.

In three dimensions, $\boldsymbol{v} + \boldsymbol{w}$ is still found a component at a time. The sum has components $v_1 + w_1$ and $v_2 + w_2$ and $v_3 + w_3$. You see how to add vectors in 4 or 5 or n dimensions. When $\boldsymbol{w}$ starts at the end of $\boldsymbol{v}$, the third side is $\boldsymbol{v} + \boldsymbol{w}$. The other way around the parallelogram is $\boldsymbol{w} + \boldsymbol{v}$. Question: Do the four sides all lie in the same plane? *Yes.* And the sum $\boldsymbol{v} + \boldsymbol{w} - \boldsymbol{v} - \boldsymbol{w}$ goes completely around to produce the _____ vector.

A typical linear combination of three vectors in three dimensions is $\boldsymbol{u} + 4\boldsymbol{v} - 2\boldsymbol{w}$:

Linear combination
Multiply by 1, 4, −2
Then add

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}.$$

The Important Questions

For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$. For two vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v}$. For three vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$. Will you take the big step from *one* combination to *all* combinations? Every c and d and e are allowed. Suppose the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in three-dimensional space:

1. What is the picture of *all* combinations $c\boldsymbol{u}$?
2. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v}$?
3. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

The answers depend on the particular vectors $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$. If they were zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors (components chosen at random), here are the three answers. This is the key to our subject:

1. The combinations $c\boldsymbol{u}$ fill a ***line***.
2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane***.
3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***three-dimensional space***.

The zero vector $(0, 0, 0)$ is on the line because c can be zero. It is on the plane because c and d can be zero. The line of vectors $c\boldsymbol{u}$ is infinitely long (forward and backward). It is the plane of all $c\boldsymbol{u} + d\boldsymbol{v}$ (combining two vectors in three-dimensional space) that I especially ask you to think about.

Adding all $c\boldsymbol{u}$ on one line to all $d\boldsymbol{v}$ on the other line fills in the plane in Figure 1.3.

When we include a third vector $\boldsymbol{w}$, the multiples $e\boldsymbol{w}$ give a third line. Suppose that third line is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$. Then combining all $e\boldsymbol{w}$ with all $c\boldsymbol{u} + d\boldsymbol{v}$ fills up the whole three-dimensional space.

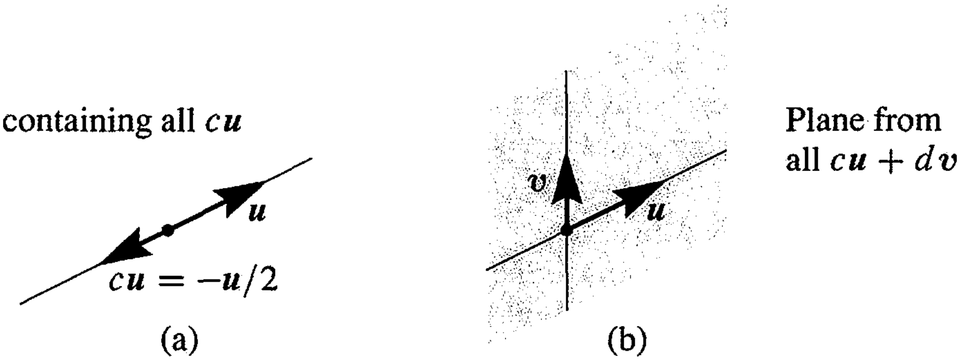


Figure 1.3: (a) Line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

This is the typical situation! **Line**, then **plane**, then **space**. But other possibilities exist. When $\boldsymbol{w}$ happens to be $c\boldsymbol{u} + d\boldsymbol{v}$, the third vector is in the plane of the first two. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ will not go outside that $\boldsymbol{uv}$ plane. We do not get the full three-dimensional space. Please think about the special cases in Problem 1.

■ REVIEW OF THE KEY IDEAS ■

1. A vector $\boldsymbol{v}$ in two-dimensional space has two components v_1 and v_2.
2. $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1,\ v_2 + w_2)$ and $c\boldsymbol{v} = (cv_1, cv_2)$ are found a component at a time.
3. A linear combination of three vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\boldsymbol{w}$ is $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$.
4. Take *all* linear combinations of $\boldsymbol{u}$, or $\boldsymbol{u}$ and $\boldsymbol{v}$, or $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. In three dimensions, those combinations typically fill a line, then a plane, and the whole space $\mathbf{R}^3$.

■ WORKED EXAMPLES ■

1.1 A The linear combinations of $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (0, 1, 1)$ fill a plane. *Describe that plane.* Find a vector that is *not* a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Solution **The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ fill a plane in $\mathbf{R}^3$.** The vectors in that plane allow any c and d. The plane of Figure 1.3 fills in between the "$\boldsymbol{u}$-line" and the "$\boldsymbol{v}$-line".

$$\text{Combinations} \quad c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c+d \\ d \end{bmatrix} \text{ fill a plane.}$$

Four particular vectors in that plane are $(0, 0, 0)$ and $(2, 3, 1)$ and $(5, 7, 2)$ and $(\pi, 2\pi, \pi)$. The second component $c + d$ is always the sum of the first and third components. *The vector* $(1, 2, 3)$ *is not in the plane, because* $2 \neq 1 + 3$.

Another description of this plane through $(0,0,0)$ is to know that $\boldsymbol{n} = (1,-1,1)$ is **perpendicular** to the plane. Section 1.2 will confirm that 90° angle by testing dot products: $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{n} = 0$.

1.1 B For $\boldsymbol{v} = (1,0)$ and $\boldsymbol{w} = (0,1)$, describe all points $c\boldsymbol{v}$ with (1) *whole numbers* c (2) *nonnegative* $c \geq 0$. Then add all vectors $d\boldsymbol{w}$ and describe all $c\boldsymbol{v} + d\boldsymbol{w}$.

Solution

(1) The vectors $c\boldsymbol{v} = (c,0)$ with whole numbers c are **equally spaced points** along the x axis (the direction of $\boldsymbol{v}$). They include $(-2,0), (-1,0), (0,0), (1,0), (2,0)$.

(2) The vectors $c\boldsymbol{v}$ with $c \geq 0$ fill a ***half-line***. It is the *positive* x axis. This half-line starts at $(0,0)$ where $c = 0$. It includes $(\pi,0)$ but not $(-\pi,0)$.

(1′) Adding all vectors $d\boldsymbol{w} = (0,d)$ puts a vertical line through those points $c\boldsymbol{v}$. We have infinitely many ***parallel lines*** from (*whole number* c, *any number* d).

(2′) Adding all vectors $d\boldsymbol{w}$ puts a vertical line through every $c\boldsymbol{v}$ on the half-line. Now we have a ***half-plane***. It is the right half of the xy plane (any $x \geq 0$, any height y).

1.1 C Find two equations for the unknowns c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution In applying mathematics, many problems have two parts:

1 *Modeling part* Express the problem by a set of equations.

2 *Computational part* Solve those equations by a fast and accurate algorithm.

Here we are only asked for the first part (the equations). Chapter 2 is devoted to the second part (the algorithm). Our example fits into a fundamental model for linear algebra:

$$\text{Find } c_1, \ldots, c_n \quad \text{so that} \quad c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{b}.$$

For $n = 2$ we could find a formula for the $\boldsymbol{c}$'s. The "elimination method" in Chapter 2 succeeds far beyond $n = 100$. For n greater than 1 million, see Chapter 9. Here $n = 2$:

Vector equation $$c\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The required equations for c and d just come from the two components separately:

Two scalar equations $$\begin{aligned} 2c - d &= 1 \\ -c + 2d &= 0 \end{aligned}$$

You could think of those as two lines that cross at the solution $c = \frac{2}{3}, d = \frac{1}{3}$.

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

1 Describe geometrically (line, plane, or all of $\mathbf{R}^3$) all linear combinations of

$$\text{(a)} \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } \begin{bmatrix}3\\6\\9\end{bmatrix} \qquad \text{(b)} \begin{bmatrix}1\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\2\\3\end{bmatrix} \qquad \text{(c)} \begin{bmatrix}2\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\2\\2\end{bmatrix} \text{ and } \begin{bmatrix}2\\2\\3\end{bmatrix}$$

2 Draw $\boldsymbol{v} = \begin{bmatrix}4\\1\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}-2\\2\end{bmatrix}$ and $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ in a single xy plane.

3 If $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix}5\\1\end{bmatrix}$ and $\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix}1\\5\end{bmatrix}$, compute and draw $\boldsymbol{v}$ and $\boldsymbol{w}$.

4 From $\boldsymbol{v} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}1\\2\end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

5 Compute $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$ and $2\boldsymbol{u} + 2\boldsymbol{v} + \boldsymbol{w}$. How do you know $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie in a plane?

$$\textbf{In a plane} \qquad \boldsymbol{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix}-3\\1\\-2\end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix}2\\-3\\-1\end{bmatrix}.$$

6 Every combination of $\boldsymbol{v} = (1, -2, 1)$ and $\boldsymbol{w} = (0, 1, -1)$ has components that add to ______. Find c and d so that $c\boldsymbol{v} + d\boldsymbol{w} = (3, 3, -6)$.

7 In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix}2\\1\end{bmatrix} + d\begin{bmatrix}0\\1\end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

8 The parallelogram in Figure 1.1 has diagonal $\boldsymbol{v} + \boldsymbol{w}$. What is its other diagonal? What is the sum of the two diagonals? Draw that vector sum.

9 If three corners of a parallelogram are $(1, 1)$, $(4, 2)$, and $(1, 3)$, what are all three of the possible fourth corners? Draw two of them.

Problems 10–14 are about special vectors on cubes and clocks in Figure 1.4.

10 Which point of the cube is $\boldsymbol{i} + \boldsymbol{j}$? Which point is the vector sum of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$? Describe all points (x, y, z) in the cube.

11 Four corners of the cube are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces are ______.

12 How many corners does a cube have in 4 dimensions? How many 3D faces? How many edges? A typical corner is $(0, 0, 1, 0)$. A typical edge goes to $(0, 1, 0, 0)$.

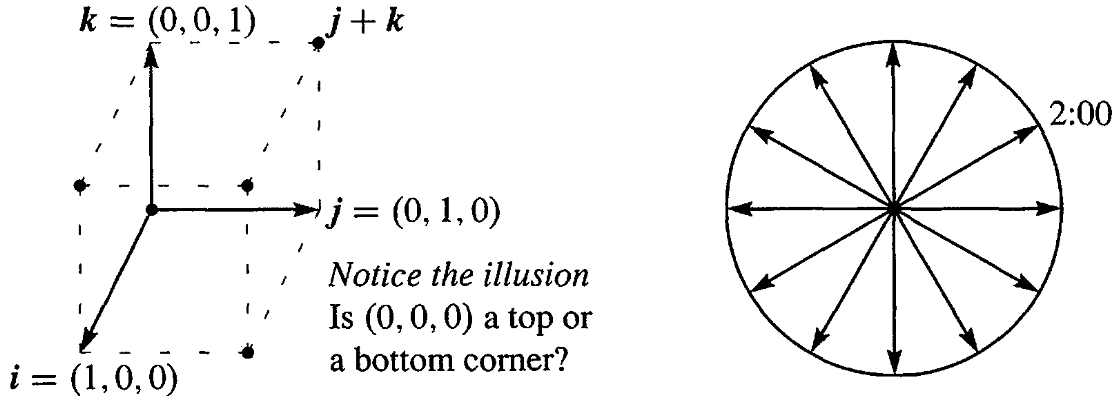


Figure 1.4: Unit cube from $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and twelve clock vectors.

13 (a) What is the sum V of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

(c) What are the components of that 2:00 vector $\boldsymbol{v} = (\cos\theta, \sin\theta)$?

14 Suppose the twelve vectors start from 6:00 at the bottom instead of $(0, 0)$ at the center. The vector to 12:00 is doubled to $(0, 2)$. Add the new twelve vectors.

Problems 15–19 go further with linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (Figure 1.5a).

15 Figure 1.5a shows $\frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. Mark the points $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$.

16 Mark the point $-\boldsymbol{v} + 2\boldsymbol{w}$ and any other combination $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$. Draw the line of all combinations that have $c + d = 1$.

17 Locate $\frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{2}{3}\boldsymbol{v} + \frac{2}{3}\boldsymbol{w}$. The combinations $c\boldsymbol{v} + c\boldsymbol{w}$ fill out what line?

18 Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

19 Restricted only by $c \ge 0$ and $d \ge 0$ draw the "cone" of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

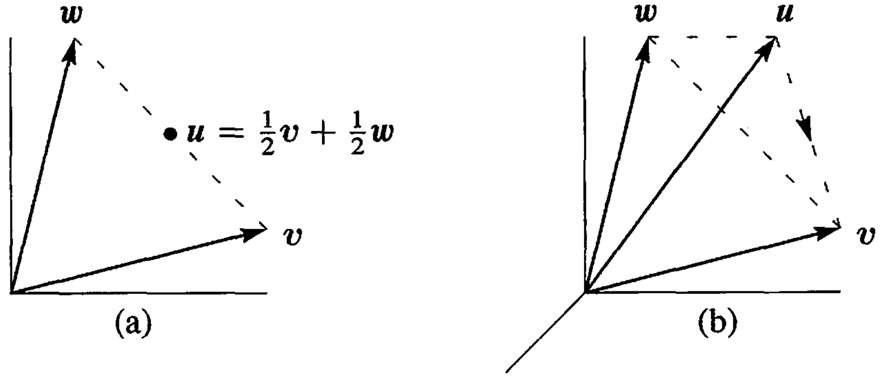


Figure 1.5: Problems **15–19** in a plane Problems **20–25** in 3-dimensional space

Problems 20–25 deal with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in three-dimensional space (see Figure 1.5b).

20 Locate $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ in Figure 1.5b. Challenge problem: Under what restrictions on c, d, e, will the combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill in the dashed triangle? To stay in the triangle, one requirement is $c \geq 0, d \geq 0, e \geq 0$.

21 The three sides of the dashed triangle are $\boldsymbol{v} - \boldsymbol{u}$ and $\boldsymbol{w} - \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{w}$. Their sum is ______. Draw the head-to-tail addition around a plane triangle of $(3, 1)$ plus $(-1, 1)$ plus $(-2, -2)$.

22 Shade in the pyramid of combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ with $c \geq 0, d \geq 0, e \geq 0$ and $c + d + e \leq 1$. Mark the vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ as inside or outside this pyramid.

23 If you look at *all* combinations of those $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$, is there any vector that can't be produced from $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$? Different answer if $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all in ______.

24 Which vectors are combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$, and *also* combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$?

25 Draw vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a line. Find vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a plane.

26 What combination $c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ produces $\begin{bmatrix} 14 \\ 8 \end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.

27 *Review Question.* In xyz space, where is the plane of all linear combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1, 1, 0)$?

Challenge Problems

28 Find vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ so that $\boldsymbol{v} + \boldsymbol{w} = (4, 5, 6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2, 5, 8)$. This is a question with ______ unknown numbers, and an equal number of equations to find those numbers.

29 Find two different combinations of the three vectors $\boldsymbol{u} = (1, 3)$ and $\boldsymbol{v} = (2, 7)$ and $\boldsymbol{w} = (1, 5)$ that produce $\boldsymbol{b} = (0, 1)$. Slightly delicate question: If I take any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane, will there always be two different combinations that produce $\boldsymbol{b} = (0, 1)$?

30 The linear combinations of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (c, d)$ fill the plane unless ______. Find four vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$ with four components each so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} + f\boldsymbol{z}$ produce all vectors (b_1, b_2, b_3, b_4) in four-dimensional space.

31 Write down three equations for c, d, e so that $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$. Can you somehow find c, d, and e?

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1.2 Lengths and Dot Products

The first section backed off from multiplying vectors. Now we go forward to define the *"dot product"* of $\boldsymbol{v}$ and $\boldsymbol{w}$. This multiplication involves the separate products $v_1 w_1$ and $v_2 w_2$, but it doesn't stop there. Those two numbers are added to produce the single number $\boldsymbol{v} \cdot \boldsymbol{w}$. *This is the geometry section (lengths and angles).*

DEFINITION The ***dot product*** or ***inner product*** of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2. \qquad (1)$$

Example 1 The vectors $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ have a *zero* dot product:

Dot product is zero
Perpendicular vectors
$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular.* The angle between them is 90°. When we drew them in Figure 1.1, we saw a rectangle (not just any parallelogram). The clearest example of perpendicular vectors is $\boldsymbol{i} = (1, 0)$ along the x axis and $\boldsymbol{j} = (0, 1)$ up the y axis. Again the dot product is $\boldsymbol{i} \cdot \boldsymbol{j} = 0 + 0 = 0$. Those vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ form a right angle.

The dot product of $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (3, 1)$ is 5. Soon $\boldsymbol{v} \cdot \boldsymbol{w}$ will reveal the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ (not 90°). Please check that $\boldsymbol{w} \cdot \boldsymbol{v}$ is also 5.

The dot product $\boldsymbol{w} \cdot \boldsymbol{v}$ equals $\boldsymbol{v} \cdot \boldsymbol{w}$. The order of $\boldsymbol{v}$ and $\boldsymbol{w}$ makes no difference.

Example 2 Put a weight of 4 at the point $x = -1$ (left of zero) and a weight of 2 at the point $x = 2$ (right of zero). The x axis will balance on the center point (like a see-saw). The weights balance because the dot product is $(4)(-1) + (2)(2) = 0$.

This example is typical of engineering and science. The vector of weights is $(w_1, w_2) = (4, 2)$. The vector of distances from the center is $(v_1, v_2) = (-1, 2)$. The weights times the distances, $w_1 v_1$ and $w_2 v_2$, give the "moments". The equation for the see-saw to balance is $w_1 v_1 + w_2 v_2 = 0$.

Example 3 Dot products enter in economics and business. We have three goods to buy and sell. Their prices are (p_1, p_2, p_3) for each unit—this is the "price vector" $\boldsymbol{p}$. The quantities we buy or sell are (q_1, q_2, q_3)—positive when we sell, negative when we buy. *Selling q_1 units at the price p_1 brings in $q_1 p_1$.* The total income (quantities q times prices p) is ***the dot product $\boldsymbol{q} \cdot \boldsymbol{p}$ in three dimensions***:

$$\textbf{\textit{Income}} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1 p_1 + q_2 p_2 + q_3 p_3 = \textbf{\textit{dot product.}}$$

A zero dot product means that "the books balance". Total sales equal total purchases if $\boldsymbol{q} \cdot \boldsymbol{p} = 0$. Then $\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$ (in three-dimensional space). A supermarket with thousands of goods goes quickly into high dimensions.

Small note: Spreadsheets have become essential in management. They compute linear combinations and dot products. What you see on the screen is a matrix.

Main point To compute $\boldsymbol{v} \cdot \boldsymbol{w}$, multiply each v_i times w_i. Then add $\Sigma v_i w_i$.

Lengths and Unit Vectors

An important case is the dot product of a vector *with itself.* In this case $\boldsymbol{v}$ equals $\boldsymbol{w}$. When the vector is $\boldsymbol{v} = (1, 2, 3)$, the dot product with itself is $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2 = 14$:

Dot product $\boldsymbol{v} \cdot \boldsymbol{v}$
Length squared

$$\|\boldsymbol{v}\|^2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = \mathbf{14}.$$

Instead of a 90° angle between vectors we have 0°. The answer is not zero because $\boldsymbol{v}$ is not perpendicular to itself. The dot product $\boldsymbol{v} \cdot \boldsymbol{v}$ gives the *length of* $\boldsymbol{v}$ *squared.*

DEFINITION The ***length*** $\|\boldsymbol{v}\|$ of a vector $\boldsymbol{v}$ is the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$:

Length = norm($\boldsymbol{v}$) **length** $= \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}.$

In two dimensions the length is $\sqrt{v_1^2 + v_2^2}$. In three dimensions it is $\sqrt{v_1^2 + v_2^2 + v_3^2}$. By the calculation above, the length of $\boldsymbol{v} = (1, 2, 3)$ is $\|\boldsymbol{v}\| = \sqrt{14}$.

Here $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ is just the ordinary length of the arrow that represents the vector. In two dimensions, the arrow is in a plane. If the components are 1 and 2, the arrow is the third side of a right triangle (Figure 1.6). The Pythagoras formula $a^2 + b^2 = c^2$, which connects the three sides, is $1^2 + 2^2 = \|\boldsymbol{v}\|^2$.

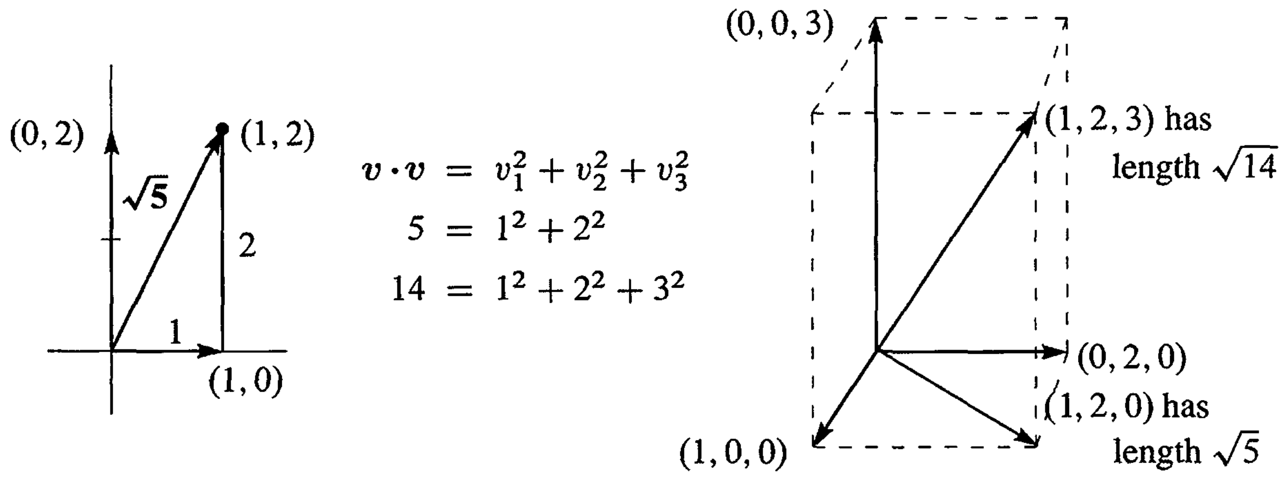
For the length of $\boldsymbol{v} = (1, 2, 3)$, we used the right triangle formula twice. The vector $(1, 2, 0)$ in the base has length $\sqrt{5}$. This base vector is perpendicular to $(0, 0, 3)$ that goes straight up. So the diagonal of the box has length $\|\boldsymbol{v}\| = \sqrt{5 + 9} = \sqrt{14}$.

The length of a four-dimensional vector would be $\sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$. Thus the vector $(1, 1, 1, 1)$ has length $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. This is the diagonal through a unit cube in four-dimensional space. The diagonal in n dimensions has length $\sqrt{n}$.

The word "**unit**" is always indicating that some measurement equals "one". The unit price is the price for one item. A unit cube has sides of length one. A unit circle is a circle with radius one. Now we define the idea of a "unit vector".

DEFINITION ***A unit vector $\boldsymbol{u}$ is a vector whose length equals one.*** Then $\boldsymbol{u} \cdot \boldsymbol{u} = \mathbf{1}$.

An example in four dimensions is $\boldsymbol{u} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $\boldsymbol{u} \cdot \boldsymbol{u}$ is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. We divided $\boldsymbol{v} = (1, 1, 1, 1)$ by its length $\|\boldsymbol{v}\| = 2$ to get this unit vector.



$$\begin{aligned} v \cdot v &= v_1^2 + v_2^2 + v_3^2 \\ 5 &= 1^2 + 2^2 \\ 14 &= 1^2 + 2^2 + 3^2 \end{aligned}$$

Figure 1.6: The length $\sqrt{v \cdot v}$ of two-dimensional and three-dimensional vectors.

Example 4 The standard unit vectors along the x and y axes are written $\boldsymbol{i}$ and $\boldsymbol{j}$. In the xy plane, the unit vector that makes an angle "theta" with the x axis is $(\cos\theta, \sin\theta)$:

$$\textbf{Unit vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

When $\theta = 0$, the horizontal vector $\boldsymbol{u}$ is $\boldsymbol{i}$. When $\theta = 90^\circ$ (or $\frac{\pi}{2}$ radians), the vertical vector is $\boldsymbol{j}$. At any angle, the components $\cos\theta$ and $\sin\theta$ produce $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ because $\cos^2\theta + \sin^2\theta = 1$. These vectors reach out to the unit circle in Figure 1.7. Thus $\cos\theta$ and $\sin\theta$ are simply the coordinates of that point at angle θ on the unit circle.

Since $(2, 2, 1)$ has length 3, the vector $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ has length 1. Check that $\boldsymbol{u} \cdot \boldsymbol{u} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$. For a unit vector, **divide any nonzero v by its length $\|v\|$.**

Unit vector $\quad \boldsymbol{u} = v/\|v\|$ **is a unit vector in the same direction as v.**

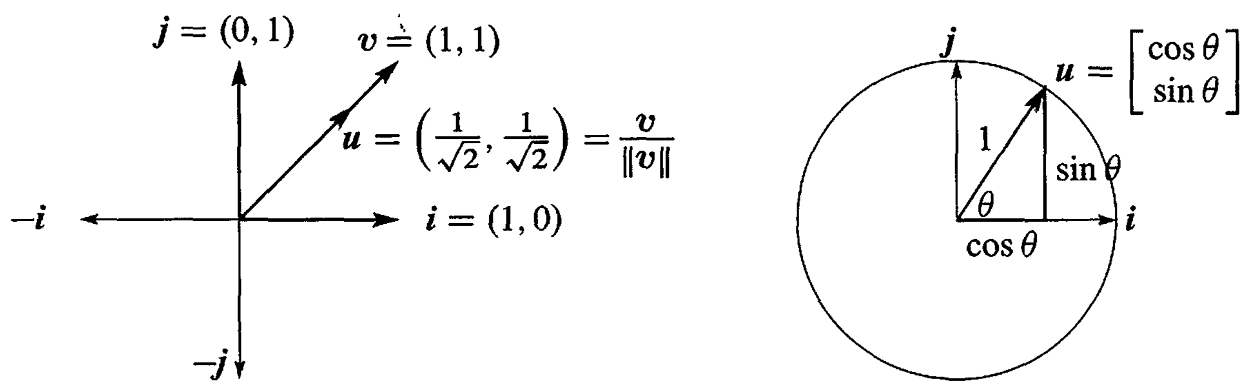


Figure 1.7: The coordinate vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. The unit vector $\boldsymbol{u}$ at angle 45° (left) divides $v = (1, 1)$ by its length $\|v\| = \sqrt{2}$. The unit vector $\boldsymbol{u} = (\cos\theta, \sin\theta)$ is at angle θ.

The Angle Between Two Vectors

We stated that perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is 90°. To explain this, we have to connect angles to dot products. Then we show how $\boldsymbol{v} \cdot \boldsymbol{w}$ finds the angle between any two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

Right angles *The dot product is* $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ *when* $\boldsymbol{v}$ *is perpendicular to* $\boldsymbol{w}$.

Proof When $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, they form two sides of a right triangle. The third side is $\boldsymbol{v} - \boldsymbol{w}$ (the hypotenuse going across in Figure 1.8). The *Pythagoras Law* for the sides of a right triangle is $a^2 + b^2 = c^2$:

$$\textbf{Perpendicular vectors} \quad \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2 \tag{2}$$

Writing out the formulas for those lengths in two dimensions, this equation is

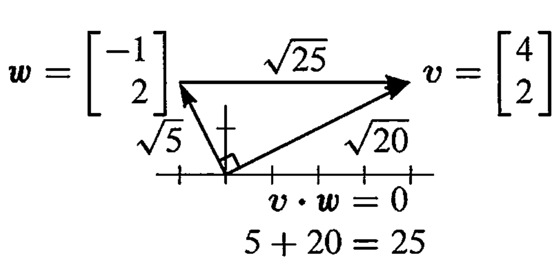
$$\textbf{Pythagoras} \qquad (v_1^2 + v_2^2) + (w_1^2 + w_2^2) = (v_1 - w_1)^2 + (v_2 - w_2)^2. \tag{3}$$

The right side begins with $v_1^2 - 2v_1w_1 + w_1^2$. Then v_1^2 and w_1^2 are on both sides of the equation and they cancel, leaving $-2v_1w_1$. Also v_2^2 and w_2^2 cancel, leaving $-2v_2w_2$. (In three dimensions there would be $-2v_3w_3$.) Now divide by -2:

$$0 = -2v_1w_1 - 2v_2w_2 \quad \text{which leads to} \quad \boldsymbol{v_1w_1 + v_2w_2 = 0}. \tag{4}$$

Conclusion Right angles produce $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is $\theta = 90°$. Then $\cos\theta = 0$. The zero vector $\boldsymbol{v} = \boldsymbol{0}$ is perpendicular to every vector $\boldsymbol{w}$ because $\boldsymbol{0} \cdot \boldsymbol{w}$ is always zero.

Now suppose $\boldsymbol{v} \cdot \boldsymbol{w}$ is **not zero.** It may be positive, it may be negative. The sign of $\boldsymbol{v} \cdot \boldsymbol{w}$ immediately tells whether we are below or above a right angle. The angle is less than 90° when $\boldsymbol{v} \cdot \boldsymbol{w}$ is positive. The angle is above 90° when $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative. The right side of Figure 1.8 shows a typical vector $\boldsymbol{v} = (3, 1)$. The angle with $\boldsymbol{w} = (1, 3)$ is less than 90° because $\boldsymbol{v} \cdot \boldsymbol{w} = 6$ is positive.



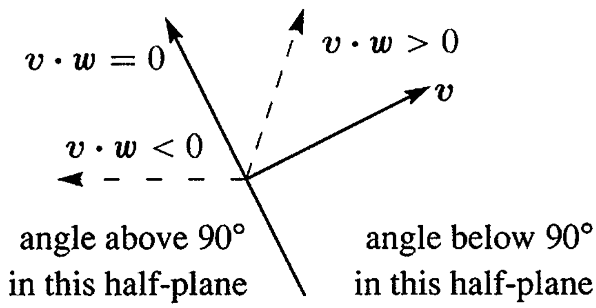


Figure 1.8: Perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. Then $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2$.

The borderline is where vectors are perpendicular to $\boldsymbol{v}$. On that dividing line between plus and minus, $(1, -3)$ is perpendicular to $(3, 1)$. The dot product is zero.

The dot product reveals the exact angle θ. This is not necessary for linear algebra—you could stop here! Once we have matrices, we won't come back to θ. But while we are on the subject of angles, this is the place for the formula.

Start with **unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$**. The sign of $\boldsymbol{u} \cdot \boldsymbol{U}$ tells whether $\theta < 90°$ or $\theta > 90°$. Because the vectors have length 1, we learn more than that. ***The dot product $\boldsymbol{u} \cdot \boldsymbol{U}$ is the cosine of θ.*** This is true in any number of dimensions.

Unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ at angle θ have $\boldsymbol{u} \cdot \boldsymbol{U} = \cos\theta$. **Certainly** $|\boldsymbol{u} \cdot \boldsymbol{U}| \leq 1$.

Remember that $\cos\theta$ is never greater than 1. It is never less than -1. ***The dot product of unit vectors is between -1 and 1.***

Figure 1.9 shows this clearly when the vectors are $\boldsymbol{u} = (\cos\theta, \sin\theta)$ and $\boldsymbol{i} = (1, 0)$. The dot product is $\boldsymbol{u} \cdot \boldsymbol{i} = \cos\theta$. That is the cosine of the angle between them.

After rotation through any angle α, these are still unit vectors. The vector $\boldsymbol{i} = (1, 0)$ rotates to $(\cos\alpha, \sin\alpha)$. The vector $\boldsymbol{u}$ rotates to $(\cos\beta, \sin\beta)$ with $\beta = \alpha + \theta$. Their dot product is $\cos\alpha\cos\beta + \sin\alpha\sin\beta$. From trigonometry this is the same as $\cos(\beta - \alpha)$. But $\beta - \alpha$ is the angle θ, so the dot product is $\cos\theta$.

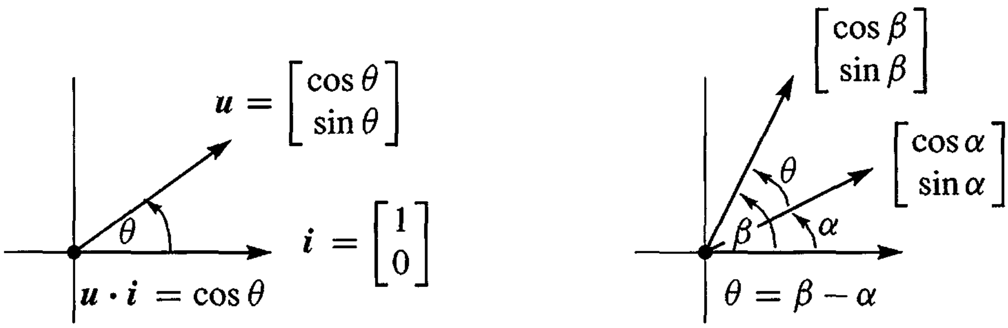


Figure 1.9: The dot product of unit vectors is the cosine of the angle θ.

Problem 24 proves $|\boldsymbol{u} \cdot \boldsymbol{U}| \leq 1$ directly, without mentioning angles. The inequality and the cosine formula $\boldsymbol{u} \cdot \boldsymbol{U} = \cos\theta$ are always true for unit vectors.

What if $\boldsymbol{v}$ and $\boldsymbol{w}$ are not unit vectors? Divide by their lengths to get $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ and $\boldsymbol{U} = \boldsymbol{w}/\|\boldsymbol{w}\|$. Then the dot product of those unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ gives $\cos\theta$.

COSINE FORMULA If $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors then $\dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \cos\theta$.

Whatever the angle, this dot product of $v/\|v\|$ with $w/\|w\|$ never exceeds one. That is the ***"Schwarz inequality"*** $|v \cdot w| \le \|v\|\,\|w\|$ for dot products—or more correctly the Cauchy-Schwarz-Buniakowsky inequality. It was found in France and Germany and Russia (and maybe elsewhere—it is the most important inequality in mathematics).

Since $|\cos\theta|$ never exceeds 1, the cosine formula gives two great inequalities:

SCHWARZ INEQUALITY	$\|v \cdot w\| \le \lVert v\rVert\,\lVert w\rVert$
TRIANGLE INEQUALITY	$\lVert v + w\rVert \le \lVert v\rVert + \lVert w\rVert$

Example 5 Find $\cos\theta$ for $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

Solution The dot product is $v \cdot w = 4$. Both v and w have length $\sqrt{5}$. The cosine is $4/5$.

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

The angle is below 90° because $v \cdot w = 4$ is positive. By the Schwarz inequality, $v \cdot w = 4$ is less than $\|v\|\,\|w\| = 5$. Side $3 = \|v + w\|$ is less than side 1 + side 2, by the triangle inequality. For $v + w = (3, 3)$ that says $\sqrt{18} < \sqrt{5} + \sqrt{5}$. Square this to get $18 < 20$.

Example 6 The dot product of $v = (a, b)$ and $w = (b, a)$ is $2ab$. Both lengths are $\sqrt{a^2 + b^2}$. The Schwarz inequality in this case says that $2ab \le a^2 + b^2$.

This is more famous if we write $x = a^2$ and $y = b^2$. The "geometric mean" $\sqrt{xy}$ is not larger than the "arithmetic mean" = average $\frac{1}{2}(x + y)$.

$$\begin{matrix}\textbf{Geometric} \\ \textbf{mean}\end{matrix} \le \begin{matrix}\textbf{Arithmetic} \\ \textbf{mean}\end{matrix} \qquad ab \le \frac{a^2 + b^2}{2} \quad \text{becomes} \quad \sqrt{xy} \le \frac{x + y}{2}.$$

Example 5 had $a = 2$ and $b = 1$. So $x = 4$ and $y = 1$. The geometric mean $\sqrt{xy} = 2$ is below the arithmetic mean $\frac{1}{2}(1 + 4) = 2.5$.

Notes on Computing

Write the components of v as $v(1), \ldots, v(N)$ and similarly for w. In FORTRAN, the sum $v + w$ requires a loop to add components separately. The dot product also uses a loop to add the separate $v(j)w(j)$. Here are VPLUSW and VDOTW:

FORTRAN

```
   DO 10 J = 1,N                    DO 10 J = 1,N
10 VPLUSW(J) = v(J) + w(J)       10 VDOTW = VDOTW + V(J) * W(J)
```

MATLAB *and also* PYTHON *work directly with whole vectors, not their components.* No loop is needed. When v and w have been defined, $v + w$ is immediately understood.

Input v and w as rows—the prime $'$ transposes them to columns. $2v + 3w$ uses $*$ for multiplication by 2 and 3. The result will be printed unless the line ends in a semicolon.

MATLAB $v = [2 \;\; 3 \;\; 4]'$; $w = [1 \;\; 1 \;\; 1]'$; $u = 2 * v + 3 * w$

The dot product $v \cdot w$ is usually seen as ***a row times a column*** (***with no dot***):

Instead of $\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}3\\4\end{bmatrix}$ we more often see $[1 \;\; 2]\begin{bmatrix}3\\4\end{bmatrix}$ or $v' * w$

The length of v is known to MATLAB as norm(v). We could define it ourselves as sqrt$(v' * v)$, using the square root function—also known. The cosine we have to define ourselves! The angle (in radians) comes from the *arc cosine* (acos) function:

Cosine formula cosine = $v' * w/$(norm$(v) *$ norm(w))
Angle formula angle = acos (cosine)

An M-file would create a new function **cosine** (v, w) for future use. The M-files created especially for this book are listed at the end. R and PYTHON are open source software.

■ REVIEW OF THE KEY IDEAS ■

1. The dot product $v \cdot w$ multiplies each component v_i by w_i and adds all $v_i w_i$.
2. The length $\|v\|$ of a vector is the square root of $v \cdot v$.
3. $u = v/\|v\|$ is a ***unit vector***. Its length is 1.
4. The dot product is $v \cdot w = 0$ when vectors v and w are perpendicular.
5. The cosine of θ (the angle between any nonzero v and w) never exceeds 1:

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} \qquad \textbf{\textit{Schwarz inequality}} \quad |v \cdot w| \le \|v\|\,\|w\|.$$

Problem 21 will produce the ***triangle inequality*** $\|v + w\| \le \|v\| + \|w\|$.

■ WORKED EXAMPLES ■

1.2 A For the vectors $v = (3, 4)$ and $w = (4, 3)$ test the Schwarz inequality on $v \cdot w$ and the triangle inequality on $\|v + w\|$. Find $\cos\theta$ for the angle between v and w. When will we have *equality* $|v \cdot w| = \|v\|\,\|w\|$ and $\|v + w\| = \|v\| + \|w\|$?

Solution The dot product is $v \cdot w = (3)(4) + (4)(3) = 24$. The length of v is $\|v\| = \sqrt{9+16} = 5$ and also $\|w\| = 5$. The sum $v + w = (7,7)$ has length $7\sqrt{2} < 10$.

Schwarz inequality	$\lvert v \cdot w\rvert \leq \|v\| \, \|w\|$ is $24 < 25$.	
Triangle inequality	$\|v + w\| \leq \|v\| + \|w\|$ is $7\sqrt{2} < 5 + 5$.	
Cosine of angle	$\cos\theta = \frac{24}{25}$	Thin angle from $v = (3,4)$ to $w = (4,3)$

Suppose one vector is a multiple of the other as in $w = cv$. Then the angle is $0°$ or $180°$. In this case $\lvert\cos\theta\rvert = 1$ and $\lvert v \cdot w\rvert$ *equals* $\|v\| \, \|w\|$. If the angle is $0°$, as in $w = 2v$, then $\|v + w\| = \|v\| + \|w\|$. The triangle is completely flat.

1.2 B Find a unit vector u in the direction of $v = (3,4)$. Find a unit vector U that is perpendicular to u. How many possibilities for U?

Solution For a unit vector u, divide v by its length $\|v\| = 5$. For a perpendicular vector V we can choose $(-4,3)$ since the dot product $v \cdot V$ is $(3)(-4) + (4)(3) = 0$. For a *unit* vector U, divide V by its length $\|V\|$:

$$u = \frac{v}{\|v\|} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad U = \frac{V}{\|V\|} = \left(-\frac{4}{5}, \frac{3}{5}\right) \qquad u \cdot U = 0$$

The only other perpendicular unit vector would be $-U = (\frac{4}{5}, -\frac{3}{5})$.

1.2 C Find a vector $x = (c,d)$ that has dot products $x \cdot r = 1$ and $x \cdot s = 0$ with the given vectors $r = (2,-1)$ and $s = (-1,2)$.

How is this question related to Example **1.1 C**, which solved $cv + dw = b = (1,0)$?

Solution Those two dot products give linear equations for c and d. Then $x = (c,d)$.

$$\begin{array}{lll} x \cdot r = 1 & 2c - d = 1 & \textbf{The same equations as} \\ x \cdot s = 0 & -c + 2d = 0 & \textbf{in Worked Example 1.1 C} \end{array}$$

The second equation makes x perpendicular to $s = (-1,2)$. So I can see the geometry: Go in the perpendicular direction $(2,1)$. When you reach $x = \frac{1}{3}(2,1)$, the dot product with $r = (2,-1)$ has the required value $x \cdot r = 1$.

Comment on n equations for $x = (x_1, \ldots, x_n)$ in n-dimensional space

Section 1.1 would start with column vectors $v_1, \ldots, v_n$. The goal is to combine them to produce a required vector $x_1 v_1 + \cdots + x_n v_n = b$. This section would start from vectors $r_1, \ldots, r_n$. Now the goal is to find x with the required dot products $x \cdot r_i = b_i$.

Soon the v's will be the columns of a matrix A, and the r's will be the rows of A. Then the (one and only) problem will be to solve $Ax = b$.

Problem Set 1.2

1 Calculate the dot products $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$ and $\boldsymbol{w} \cdot \boldsymbol{v}$:

$$\boldsymbol{u} = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

2 Compute the lengths $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ and $\|\boldsymbol{w}\|$ of those vectors. Check the Schwarz inequalities $|\boldsymbol{u} \cdot \boldsymbol{v}| \le \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

3 Find unit vectors in the directions of $\boldsymbol{v}$ and $\boldsymbol{w}$ in Problem 1, and the cosine of the angle θ. Choose vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ that make $0°$, $90°$, and $180°$ angles with $\boldsymbol{w}$.

4 For any *unit* vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, find the dot products (actual numbers) of

(a) $\boldsymbol{v}$ and $-\boldsymbol{v}$ (b) $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ (c) $\boldsymbol{v} - 2\boldsymbol{w}$ and $\boldsymbol{v} + 2\boldsymbol{w}$

5 Find unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in the directions of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (2, 1, 2)$. Find unit vectors $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ that are perpendicular to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

6 (a) Describe every vector $\boldsymbol{w} = (w_1, w_2)$ that is perpendicular to $\boldsymbol{v} = (2, -1)$.

(b) The vectors that are perpendicular to $\boldsymbol{V} = (1, 1, 1)$ lie on a _____.

(c) The vectors that are perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a _____.

7 Find the angle θ (from its cosine) between these pairs of vectors:

(a) $\boldsymbol{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\boldsymbol{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(c) $\boldsymbol{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ (d) $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

8 True or false (give a reason if true or a counterexample if false):

(a) If $\boldsymbol{u}$ is perpendicular (in three dimensions) to $\boldsymbol{v}$ and $\boldsymbol{w}$, those vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

(b) If $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v} + 2\boldsymbol{w}$.

(c) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular unit vectors then $\|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{2}$.

9 The slopes of the arrows from $(0, 0)$ to (v_1, v_2) and (w_1, w_2) are v_2/v_1 and w_2/w_1. **Suppose the product $v_2 w_2 / v_1 w_1$ of those slopes is** -1. Show that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the vectors are perpendicular.

10 Draw arrows from $(0, 0)$ to the points $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (-2, 1)$. Multiply their slopes. That answer is a signal that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the arrows are _____.

11 If $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative, what does this say about the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$? Draw a 3-dimensional vector $\boldsymbol{v}$ (an arrow), and show where to find all $\boldsymbol{w}$'s with $\boldsymbol{v} \cdot \boldsymbol{w} < 0$.

12 With $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (1, 5)$ choose a number c so that $\boldsymbol{w} - c\boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$. Then find the formula that gives this number c for any nonzero $\boldsymbol{v}$ and $\boldsymbol{w}$. (*Note*: $c\boldsymbol{v}$ is the "projection" of $\boldsymbol{w}$ onto $\boldsymbol{v}$.)

13 Find two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ that are perpendicular to $(1, 0, 1)$ and to each other.

14 Find nonzero vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that are perpendicular to $(1, 1, 1, 1)$ and to each other.

15 The geometric mean of $x = 2$ and $y = 8$ is $\sqrt{xy} = 4$. The arithmetic mean is larger: $\frac{1}{2}(x + y) =$ _____. This would come in Example 6 from the Schwarz inequality for $\boldsymbol{v} = (\sqrt{2}, \sqrt{8})$ and $\boldsymbol{w} = (\sqrt{8}, \sqrt{2})$. Find $\cos\theta$ for this $\boldsymbol{v}$ and $\boldsymbol{w}$.

16 **How long is the vector $\boldsymbol{v} = (1, 1, \ldots, 1)$ in 9 dimensions**? Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$ and a unit vector $\boldsymbol{w}$ that is perpendicular to $\boldsymbol{v}$.

17 What are the cosines of the angles α, β, θ between the vector $(1, 0, -1)$ and the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ along the axes? Check the formula $\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1$.

Problems 18–31 lead to the main facts about lengths and angles in triangles.

18 The parallelogram with sides $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ is a rectangle. Check the Pythagoras formula $a^2 + b^2 = c^2$ which is for ***right triangles only***:

$$(\text{length of } \boldsymbol{v})^2 + (\text{length of } \boldsymbol{w})^2 = (\text{length of } \boldsymbol{v} + \boldsymbol{w})^2.$$

19 (Rules for dot products) These equations are simple but useful:

(1) $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ (2) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ (3) $(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$

Use (2) with $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ to prove $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$.

20 The "Law of Cosines" comes from $(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$:

Cosine Law $$\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2.$$

If $\theta < 90°$ show that $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$ is larger than $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ (the third side).

21 The ***triangle inequality*** says: (length of $\boldsymbol{v} + \boldsymbol{w}$) $\leq$ (length of $\boldsymbol{v}$) + (length of $\boldsymbol{w}$).

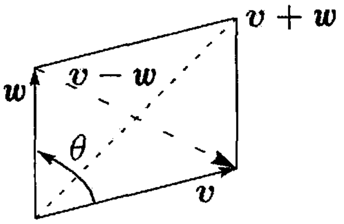
Problem 19 found $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$. Use the Schwarz inequality $\boldsymbol{v} \cdot \boldsymbol{w} \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ to show that $\|$**side 3**$\|$ can not exceed $\|$**side 1**$\|$ + $\|$**side 2**$\|$:

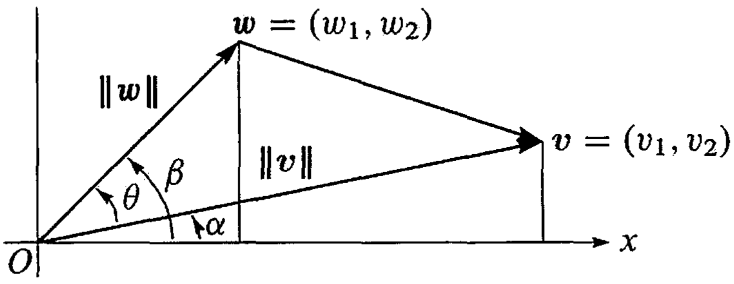
Triangle inequality $$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2 \quad \textbf{or} \quad \|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

22 The Schwarz inequality $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ by algebra instead of trigonometry:

(a) Multiply out both sides of $(v_1w_1 + v_2w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Show that the difference between those two sides equals $(v_1w_2 - v_2w_1)^2$. This cannot be negative since it is a square—so the inequality is true.





23 The figure shows that $\cos\alpha = v_1/\|v\|$ and $\sin\alpha = v_2/\|v\|$. Similarly $\cos\beta$ is _____ and $\sin\beta$ is _____. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos\beta\cos\alpha + \sin\beta\sin\alpha$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

24 One-line proof of the Schwarz inequality $|\boldsymbol{u} \cdot \boldsymbol{U}| \le 1$ for unit vectors:

$$|\boldsymbol{u} \cdot \boldsymbol{U}| \le |u_1|\,|U_1| + |u_2|\,|U_2| \le \frac{u_1^2 + U_1^2}{2} + \frac{u_2^2 + U_2^2}{2} = \frac{1+1}{2} = 1.$$

Put $(u_1, u_2) = (.6, .8)$ and $(U_1, U_2) = (.8, .6)$ in that whole line and find $\cos\theta$.

25 Why is $|\cos\theta|$ never greater than 1 in the first place?

26 If $\boldsymbol{v} = (1, 2)$ draw all vectors $\boldsymbol{w} = (x, y)$ in the xy plane with $\boldsymbol{v} \cdot \boldsymbol{w} = x + 2y = 5$. Which is the shortest $\boldsymbol{w}$?

27 (*Recommended*) If $||\boldsymbol{v}|| = 5$ and $||\boldsymbol{w}|| = 3$, what are the smallest and largest values of $||\boldsymbol{v} - \boldsymbol{w}||$? What are the smallest and largest values of $\boldsymbol{v} \cdot \boldsymbol{w}$?

Challenge Problems

28 Can three vectors in the xy plane have $\boldsymbol{u} \cdot \boldsymbol{v} < 0$ and $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ and $\boldsymbol{u} \cdot \boldsymbol{w} < 0$? I don't know how many vectors in xyz space can have all negative dot products. (Four of those vectors in the plane would certainly be impossible . . .).

29 Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $\boldsymbol{v} = (x, y, z)$ and the vector $\boldsymbol{w} = (z, x, y)$. Challenge question: Explain why $\boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ is always $-\frac{1}{2}$.

30 How could you prove $\sqrt[3]{xyz} \le \frac{1}{3}(x+y+z)$ (geometric mean $\le$ arithmetic mean)?

31 Find four perpendicular unit vectors with all components equal to $\frac{1}{2}$ or $-\frac{1}{2}$.

32 Using $\boldsymbol{v} =$ randn(3, 1) in MATLAB, create a random unit vector $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$. Using $V =$ randn(3, 30) create 30 more random unit vectors $\boldsymbol{U}_j$. What is the average size of the dot products $|\boldsymbol{u} \cdot \boldsymbol{U}_j|$? In calculus, the average $\int_0^{\pi} |\cos\theta|\,d\theta/\pi = 2/\pi$.

1.3 Matrices

This section is based on two carefully chosen examples. They both start with three vectors. I will take their combinations using *matrices*. The three vectors in the first example are $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$:

First example $$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Their linear combinations in three-dimensional space are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$:

Combinations $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \qquad (1)$$

Now something important: *Rewrite that combination using a matrix.* The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ go into the columns of the matrix A. That matrix "multiplies" a vector:

Same combination is now A times $\boldsymbol{x}$ $$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \qquad (2)$$

The numbers c, d, e are the components of a vector $\boldsymbol{x}$. The matrix A times the vector $\boldsymbol{x}$ is the same as the combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ of the three columns:

Matrix times vector $$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}. \qquad (3)$$

This is more than a definition of $A\boldsymbol{x}$, because the rewriting brings a crucial change in viewpoint. At first, the numbers c, d, e were multiplying the vectors. Now the matrix is multiplying those numbers. **The matrix A acts on the vector $\boldsymbol{x}$.** The result $A\boldsymbol{x}$ is a combination $\boldsymbol{b}$ of the columns of A.

To see that action, I will write x_1, x_2, x_3 instead of c, d, e. I will write b_1, b_2, b_3 for the components of $A\boldsymbol{x}$. With new letters we see

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}. \qquad (4)$$

The input is $\boldsymbol{x}$ and the output is $\boldsymbol{b} = A\boldsymbol{x}$. This A is a "**difference matrix**" because $\boldsymbol{b}$ contains differences of the input vector $\boldsymbol{x}$. The top difference is $x_1 - x_0 = x_1 - 0$.

Here is an example to show differences of numbers (squares in $\boldsymbol{x}$, odd numbers in $\boldsymbol{b}$):

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \text{squares} \qquad A\boldsymbol{x} = \begin{bmatrix} 1-0 \\ 4-1 \\ 9-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \boldsymbol{b}. \tag{5}$$

That pattern would continue for a 4 by 4 difference matrix. The next square would be $x_4 = 16$. The next difference would be $x_4 - x_3 = 16 - 9 = 7$ (this is the next odd number). The matrix finds all the differences at once.

Important Note. You may already have learned about multiplying $A\boldsymbol{x}$, a matrix times a vector. Probably it was explained differently, using the rows instead of the columns. The usual way takes the dot product of each row with $\boldsymbol{x}$:

Dot products with rows
$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1, x_2, x_3) \\ (-1,1,0) \cdot (x_1, x_2, x_3) \\ (0,-1,1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

Those dot products are the same x_1 and $x_2 - x_1$ and $x_3 - x_2$ that we wrote in equation (4). The new way is to work with $A\boldsymbol{x}$ *a column at a time*. Linear combinations are the key to linear algebra, and the output $A\boldsymbol{x}$ is a linear combination of the **columns** of A.

With numbers, you can multiply $A\boldsymbol{x}$ either way (I admit to using rows). With letters, columns are the good way. Chapter 2 will repeat these rules of matrix multiplication, and explain the underlying ideas. There we will multiply matrices both ways.

Linear Equations

One more change in viewpoint is crucial. Up to now, the numbers x_1, x_2, x_3 were known (called c, d, e at first). The right hand side $\boldsymbol{b}$ was not known. We found that vector of differences by multiplying $A\boldsymbol{x}$. **Now we think of $\boldsymbol{b}$ as known and we look for $\boldsymbol{x}$.**

Old question: Compute the linear combination $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$ to find $\boldsymbol{b}$.
New question: Which combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ produces a particular vector $\boldsymbol{b}$?

This is the inverse problem—to find the input $\boldsymbol{x}$ that gives the desired output $\boldsymbol{b} = A\boldsymbol{x}$. You have seen this before, as a system of linear equations for x_1, x_2, x_3. The right hand sides of the equations are b_1, b_2, b_3. We can solve that system to find x_1, x_2, x_3:

$$A\boldsymbol{x} = \boldsymbol{b} \quad \begin{aligned} x_1 \qquad\quad &= b_1 \\ -x_1 + x_2 \quad &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned} \qquad \textbf{Solution} \quad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3. \end{aligned} \tag{6}$$

Let me admit right away—most linear systems are not so easy to solve. In this example, the first equation decided $x_1 = b_1$. Then the second equation produced $x_2 = b_1 + b_2$. *The equations could be solved in order* (top to bottom) *because the matrix A was selected to be* **lower triangular**.

Look at two specific choices 0, 0, 0 and 1, 3, 5 of the right sides b_1, b_2, b_3:

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 1 \\ 1+3 \\ 1+3+5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

The first solution (all zeros) is more important than it looks. In words: *If the output is $\boldsymbol{b} = \mathbf{0}$, then the input must be $\boldsymbol{x} = \mathbf{0}$.* That statement is true for this matrix A. It is not true for all matrices. Our second example will show (for a different matrix C) how we can have $C\boldsymbol{x} = \mathbf{0}$ when $C \neq 0$ and $\boldsymbol{x} \neq \mathbf{0}$.

This matrix A is "**invertible**". From $\boldsymbol{b}$ we can recover $\boldsymbol{x}$.

The Inverse Matrix

Let me repeat the solution $\boldsymbol{x}$ in equation (6). A sum matrix will appear!

$$A\boldsymbol{x} = \boldsymbol{b} \text{ is solved by } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad (7)$$

If the differences of the x's are the b's, the sums of the b's are the x's. That was true for the odd numbers $\boldsymbol{b} = (1, 3, 5)$ and the squares $\boldsymbol{x} = (1, 4, 9)$. It is true for all vectors. **The sum matrix S in equation (7) is the inverse of the difference matrix A.**

Example: The differences of $\boldsymbol{x} = (1, 2, 3)$ are $\boldsymbol{b} = (1, 1, 1)$. So $\boldsymbol{b} = A\boldsymbol{x}$ and $\boldsymbol{x} = S\boldsymbol{b}$:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \text{ and } S\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$$

Equation (7) for the solution vector $\boldsymbol{x} = (x_1, x_2, x_3)$ tells us two important facts:

1. For every $\boldsymbol{b}$ there is one solution to $A\boldsymbol{x} = \boldsymbol{b}$. **2.** A matrix S produces $\boldsymbol{x} = S\boldsymbol{b}$.

The next chapters ask about other equations $A\boldsymbol{x} = \boldsymbol{b}$. Is there a solution? How is it computed? In linear algebra, the notation for the "inverse matrix" is A^{-1}:

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is solved by} \quad \boldsymbol{x} = A^{-1}\boldsymbol{b} = S\boldsymbol{b}.$$

Note on calculus. Let me connect these special matrices A and S to calculus. The vector $\boldsymbol{x}$ changes to a function $x(t)$. The differences $A\boldsymbol{x}$ become the *derivative* $dx/dt = b(t)$. In the inverse direction, the sum $S\boldsymbol{b}$ becomes the *integral* of $b(t)$. The Fundamental Theorem of Calculus says that ***integration S is the inverse of differentiation A***.

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \textit{ and } \boldsymbol{x} = \boldsymbol{S}\boldsymbol{b} \qquad \frac{dx}{dt} = b \text{ and } x(t) = \int_0^t b. \qquad (8)$$

The derivative of distance traveled (x) is the velocity (b). The integral of $b(t)$ is the distance $x(t)$. Instead of adding $+C$, I measured the distance from $x(0) = 0$. In the same way, the differences started at $x_0 = 0$. This zero start makes the pattern complete, when we write $x_1 - x_0$ for the first component of Ax (we just wrote x_1).

Notice another analogy with calculus. The differences of squares $0, 1, 4, 9$ are odd numbers $1, 3, 5$. The derivative of $x(t) = t^2$ is $2t$. A perfect analogy would have produced the even numbers $b = 2, 4, 6$ at times $t = 1, 2, 3$. But differences are not the same as derivatives, and our matrix A produces not $2t$ but $2t - 1$ (these one-sided "backward differences" are centered at $t - \frac{1}{2}$):

$$x(t) - x(t-1) = t^2 - (t-1)^2 = t^2 - (t^2 - 2t + 1) = 2t - 1. \tag{9}$$

The Problem Set will follow up to show that "forward differences" produce $2t + 1$. A better choice (not always seen in calculus courses) is a **centered difference** that uses $x(t+1) - x(t-1)$. Divide Δx by the distance Δt from $t - 1$ to $t + 1$, which is 2:

$$\textbf{Centered difference of } x(t) = t^2 \qquad \frac{(t+1)^2 - (t-1)^2}{2} = 2t \quad \text{exactly.} \tag{10}$$

Difference matrices are great. Centered is best. Our second example is *not invertible.*

Cyclic Differences

This example keeps the same columns $\boldsymbol{u}$ and $\boldsymbol{v}$ but changes $\boldsymbol{w}$ to a new vector $\boldsymbol{w}^*$:

$$\textbf{Second example} \qquad \boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \boldsymbol{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Now the linear combinations of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}^*$ lead to a **cyclic difference matrix** C:

$$\textbf{Cyclic} \qquad C\boldsymbol{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}. \tag{11}$$

This matrix C is not triangular. It is not so simple to solve for $\boldsymbol{x}$ when we are given $\boldsymbol{b}$. Actually it is impossible to find *the* solution to $C\boldsymbol{x} = \boldsymbol{b}$, because the three equations either have **infinitely many solutions** or else **no solution**:

$$\begin{array}{l} C\boldsymbol{x} = \boldsymbol{0} \\ \textbf{Infinitely} \\ \textbf{many } \boldsymbol{x} \end{array} \qquad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is solved by all vectors } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}. \tag{12}$$

Every constant vector (c, c, c) has zero differences when we go cyclically. This undetermined constant c is like the $+C$ that we add to integrals. The cyclic differences have $x_1 - x_3$ in the first component, instead of starting from $x_0 = 0$.

The other very likely possibility for $Cx = b$ is **no solution** at all:

$$Cx = b \quad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \begin{array}{l} \text{Left sides add to 0} \\ \text{Right sides add to 9} \\ \textit{No solution } x_1, x_2, x_3 \end{array} \tag{13}$$

Look at this example geometrically. No combination of $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}^*$ will produce the vector $\boldsymbol{b} = (1, 3, 5)$. The combinations don't fill the whole three-dimensional space. The right sides must have $b_1 + b_2 + b_3 = 0$ to allow a solution to $Cx = b$, because the left sides $x_1 - x_3$, $x_2 - x_1$, and $x_3 - x_2$ always add to zero.

Put that in different words. **All linear combinations** $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}^* = \boldsymbol{b}$ **lie on the plane given by** $b_1 + b_2 + b_3 = 0$. This subject is suddenly connecting algebra with geometry. Linear combinations can fill all of space, or only a plane. We need a picture to show the crucial difference between $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ (the first example) and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$.

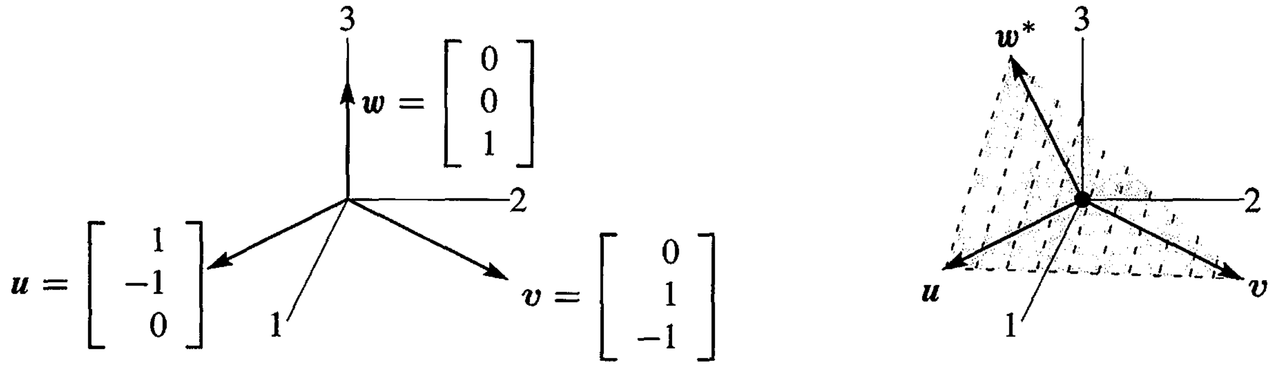


Figure 1.10: Independent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Dependent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ in a plane.

Independence and Dependence

Figure 1.10 shows those column vectors, first of the matrix A and then of C. The first two columns $\boldsymbol{u}$ and $\boldsymbol{v}$ are the same in both pictures. If we only look at the combinations of those two vectors, we will get a two-dimensional plane. **The key question is whether the third vector is in that plane:**

Independence $\boldsymbol{w}$ is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$.
Dependence $\boldsymbol{w}^*$ *is* in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$.

The important point is that the new vector $\boldsymbol{w}^*$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}^* = 0 \qquad \boldsymbol{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\boldsymbol{u} - \boldsymbol{v}. \tag{14}$$

All three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ have components adding to zero. Then all their combinations will have $b_1 + b_2 + b_3 = 0$ (as we saw above, by adding the three equations). This is the equation for the plane containing all combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$. By including $\boldsymbol{w}^*$ we get *no new vectors* because $\boldsymbol{w}^*$ is already on that plane.

The original $\boldsymbol{w} = (0, 0, 1)$ is not on the plane: $0 + 0 + 1 \neq 0$. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ fill the whole three-dimensional space. We know this already, because the solution $\boldsymbol{x} = S\boldsymbol{b}$ in equation (6) gave the right combination to produce any $\boldsymbol{b}$.

The two matrices A and C, with third columns $\boldsymbol{w}$ and $\boldsymbol{w}^*$, allowed me to mention two key words of linear algebra: independence and dependence. The first half of the course will develop these ideas much further—I am happy if you see them early in the two examples:

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are **independent**. No combination except $0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w} = \mathbf{0}$ gives $\boldsymbol{b} = \mathbf{0}$.

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ are **dependent**. Other combinations (specifically $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}^*$) give $\boldsymbol{b} = \mathbf{0}$.

You can picture this in three dimensions. The three vectors lie in a plane or they don't. Chapter 2 has n vectors in n-dimensional space. *Independence or dependence* is the key point. The vectors go into the columns of an n by n matrix:

Independent columns: $A\boldsymbol{x} = \mathbf{0}$ has one solution. A is an **invertible matrix.**

Dependent columns: $A\boldsymbol{x} = \mathbf{0}$ has many solutions. A is a **singular matrix.**

Eventually we will have n vectors in m-dimensional space. The matrix A with those n columns is now *rectangular* (m by n). Understanding $A\boldsymbol{x} = \boldsymbol{b}$ is the problem of Chapter 3.

■ REVIEW OF THE KEY IDEAS ■

1. **Matrix times vector**: $A\boldsymbol{x}$ = combination of the columns of A.
2. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$, when A is an invertible matrix.
3. The difference matrix A is inverted by the sum matrix $S = A^{-1}$.
4. The cyclic matrix C has no inverse. Its three columns lie in the same plane. Those dependent columns add to the zero vector. $C\boldsymbol{x} = \mathbf{0}$ has many solutions.
5. This section is looking ahead to key ideas, not fully explained yet.

■ WORKED EXAMPLES ■

1.3 A Change the southwest entry a_{31} of A (row 3, column 1) to $a_{31} = \mathbf{1}$:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \mathbf{1} & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ \boldsymbol{x_1} - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the solution $\boldsymbol{x}$ for any $\boldsymbol{b}$. From $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ read off the inverse matrix A^{-1}.

Solution Solve the (linear triangular) system $Ax = b$ from top to bottom:

$$\begin{array}{ll} \text{first} & x_1 = b_1 \\ \text{then} & x_2 = b_1 + b_2 \\ \text{then} & x_3 = \quad\; b_2 + b_3 \end{array} \qquad \text{This says that } x = A^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This is good practice to see the columns of the inverse matrix multiplying b_1, b_2, and b_3. The first column of A^{-1} is the solution for $b = (1,0,0)$. The second column is the solution for $b = (0,1,0)$. The third column x of A^{-1} is the solution for $Ax = b = (0,0,1)$.

The three columns of A are still independent. They don't lie in a plane. The combinations of those three columns, using the right weights x_1, x_2, x_3, can produce any three-dimensional vector $b = (b_1, b_2, b_3)$. Those weights come from $x = A^{-1}b$.

1.3 B This E is an **elimination matrix**. E has a subtraction, E^{-1} has an addition.

$$Ex = b \quad \begin{bmatrix} 1 & 0 \\ -\ell & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 \\ -\ell & 1 \end{bmatrix}$$

The first equation is $x_1 = b_1$. The second equation is $x_2 - \ell x_1 = b_2$. The inverse will *add* $\ell x_1 = \ell b_1$, because the elimination matrix *subtracted* ℓx_1:

$$x = E^{-1}b \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ \ell b_1 + b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$$

1.3 C Change C from a cyclic difference to a **centered difference** producing $x_3 - x_1$:

$$Cx = b \quad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ 0 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{15}$$

Show that $Cx = b$ can only be solved when $b_1 + b_3 = 0$. That is a plane of vectors b in three-dimensional space. Each column of C is in the plane, the matrix has no inverse. So this plane contains all combinations of those columns (which are all the vectors Cx).

Solution The first component of $b = Cx$ is x_2, and the last component of b is $-x_2$. So we always have $b_1 + b_3 = 0$, for every choice of x.

If you draw the column vectors in C, the first and third columns fall on the same line. In fact (column 1) = −(column 3). So the three columns will lie in a plane, and C is *not* an invertible matrix. We cannot solve $Cx = b$ unless $b_1 + b_3 = 0$.

I included the zeros so you could see that this matrix produces "centered differences". Row i of Cx is x_{i+1} (*right of center*) minus x_{i-1} (*left of center*). Here is the 4 by 4 centered difference matrix:

$$Cx = b \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ x_4 - x_2 \\ 0 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{16}$$

Surprisingly this matrix is now invertible! The first and last rows give x_2 and x_3. Then the middle rows give x_1 and x_4. It is possible to write down the inverse matrix C^{-1}. But 5 by 5 will be singular (*not invertible*) again ...

Problem Set 1.3

1 Find the linear combination $2s_1 + 3s_2 + 4s_3 = b$. Then write b as a matrix-vector multiplication Sx. Compute the dot products (row of S) $\cdot\, x$:

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad s_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ go into the columns of } S.$$

2 Solve these equations $Sy = b$ with s_1, s_2, s_3 in the columns of S:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

The sum of the first n odd numbers is _____.

3 Solve these three equations for y_1, y_2, y_3 in terms of B_1, B_2, B_3:

$$Sy = B \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

Write the solution y as a matrix $A = S^{-1}$ times the vector B. Are the columns of S independent or dependent?

4 Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent) (dependent). The three vectors lie in a _____. The matrix W with those columns is *not invertible.*

5 The rows of that matrix W produce three vectors (I write them as columns):

$$r_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \quad r_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \quad r_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Linear algebra says that these vectors must also lie in a plane. There must be many combinations with $y_1 r_1 + y_2 r_2 + y_3 r_3 = \mathbf{0}$. Find two sets of y's.

6 Which values of c give dependent columns (combination equals zero)?

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$$

7 If the columns combine into $A\boldsymbol{x} = \mathbf{0}$ then each row has $\boldsymbol{r} \cdot \boldsymbol{x} = 0$:

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{By rows} \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \boldsymbol{r}_3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The three rows also lie in a plane. Why is that plane perpendicular to $\boldsymbol{x}$?

8 Moving to a 4 by 4 difference equation $A\boldsymbol{x} = \boldsymbol{b}$, find the four components x_1, x_2, x_3, x_4. Then write this solution as $\boldsymbol{x} = S\boldsymbol{b}$ to find the inverse matrix $S = A^{-1}$:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \boldsymbol{b}.$$

9 What is the *cyclic* 4 by 4 difference matrix C? It will have 1 and -1 in each row. Find all solutions $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ to $C\boldsymbol{x} = \mathbf{0}$. The four columns of C lie in a "three-dimensional hyperplane" inside four-dimensional space.

10 A *forward* difference matrix Δ is *upper* triangular:

$$\Delta \boldsymbol{z} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}.$$

Find z_1, z_2, z_3 from b_1, b_2, b_3. What is the inverse matrix in $\boldsymbol{z} = \Delta^{-1}\boldsymbol{b}$?

11 Show that the forward differences $(t+1)^2 - t^2$ are $2t+1$ = *odd numbers*. As in calculus, the difference $(t+1)^n - t^n$ will begin with the derivative of t^n, which is ______.

12 The last lines of the Worked Example say that the 4 by 4 centered difference matrix in (16) *is* invertible. Solve $C\boldsymbol{x} = (b_1, b_2, b_3, b_4)$ to find its inverse in $\boldsymbol{x} = C^{-1}\boldsymbol{b}$.

Challenge Problems

13 The very last words say that the 5 by 5 centered difference matrix *is not* invertible. Write down the 5 equations $C\boldsymbol{x} = \boldsymbol{b}$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero? (The 5 columns lie on a "4-dimensional hyperplane" in 5-dimensional space.)

14 If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important; two columns are falling on one line. You could use numbers first to see how a, b, c, d are related. The question will lead to:

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent columns when it has dependent rows.